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Brief Paper

Dynamic Boundary Control of the Timoshenko Beam*

ÖMER MORGÜL†

Key Words—Distributed parameter systems; partial differential equations; boundary-value problems; stability; Lyapunov methods.

Abstract—We consider a clamped-free Timoshenko beam. To stabilize the beam vibrations, we propose a dynamic boundary control law applied at the free end of the beam. We prove that with the proposed control law, the beam vibrations uniformly and exponentially decay to zero. The proof uses a Lyapunov functional based on the energy of the system.

1. *Introduction*

In this paper, we study the uniform stabilization of the clamped-free Timoshenko beam with *dynamic* boundary control. The Timoshenko beam model is a linear beam model which accounts for both the rotatory inertia of the beam cross-sections and the deflection due to shear effects. This model is a more accurate one than both the Euler–Bernoulli beam model which yields good results when the cross-sectional dimensions of the beam are small in comparison with the length of the beam, and the Rayleigh beam model in which only the rotatory inertia of the beam cross-sections are taken into account, see Meirovitch (1967). Assuming that the beam is homogeneous with uniform cross-sections, the equations of motion of the Timoshenko beam is described by the following set of equations: for $x \in (0, L)$

$$\rho \frac{\partial^2 u}{\partial t^2} - K\left(\frac{\partial^2 u}{\partial x^2} - \frac{\partial \phi}{\partial x}\right) = 0, \tag{1}$$

$$I_\rho \frac{\partial^2 \phi}{\partial t^2} - EI\frac{\partial^2 \phi}{\partial x^2} + K\left(\phi - \frac{\partial u}{\partial x}\right) = 0, \tag{2}$$

where L is the length of the beam, t is the time variable, x is the space coordinate along the beam in its equilibrium position, $u(x, t)$ is the deflection of the beam from its equilibrium position, which is characterized by $u = 0$, $\phi(x, t)$ is the angle of rotation of the beam cross-sections due to bending, for more precise definitions, see Meirovitch (1967). The coefficients ρ, I_ρ and EI are the mass per unit length, the mass moment of inertia of the beam cross-sections and the flexural rigidity of the beam, respectively. The coefficient K is equal to kGA where G is the shear modulus, A is the cross-sectional area and k is a numerical factor depending on the shape of the beam cross-sections. All coefficients are assumed to be constant.

Equations (1) and (2) can be obtained through Hamilton's principle by using the natural energy of the beam given by:

$$E_B(t) = \frac{1}{2}\int_0^L \{\rho u_t^2 + I_\rho \phi_t^2 + K(\phi - u_x)^2 + EI\phi_x^2\}\,dx, \tag{3}$$

where a subscript letter denotes the partial differential with respect to the corresponding letter, e.g. $u_t = \partial u/\partial t$. In (3), the first two terms in the integral represent the kinetic energy due to translation and rotation and the last two terms represent the potential energy due to shearing deformation and bending, respectively, see Meirovich (1967).

The boundary conditions we have are: $t \geq 0$

$$u(0, t) = 0, \quad \phi(0, t) = 0, \tag{4}$$

$$K(\phi(L, t) - u_x(L, t)) = f_1(t), \quad -EI\phi_x(L, t) = f_2(t), \tag{5}$$

where (4) is the boundary conditions at the clamped end, (5) gives the boundary conditions at the free end, $f_1(t)$ and $f_2(t)$ are the boundary control force and the boundary control moment.

Our aim in this paper is to find appropriate control laws for $f_1(t)$ and $f_2(t)$ so that the energy $E_B(t)$ given by (3) asymptotically and uniformly decays to zero.

In recent years, the boundary control of systems described by partial differential equations has become an important research area. Chen (1979) established the uniform stabilization of the wave equation in any space dimension with the boundary control. Recently, Chen *et al.* (1987) established the uniform stabilization of the Euler–Bernoulli beam and Kim and Renardy (1987) obtained similar results for the Timoshenko beam. Recently these results have been extended to the rotating flexible structures, see Morgül (1990, 1991a,b). A good source of references to papers in which the boundary control techniques are treated can be found in Lagnese (1989). The stabilizing control laws presented here are more general than that of Kim and Renardy (1987), hence our results could be interpreted as a generalization of the results presented therein.

This paper is organized as follows. In Section 2, we propose a stabilizing control law and show that the system under investigation is well-posed, i.e. there exists a solution and this solution is unique in an appropriate space. In the Section 3 we show that the solutions decay exponentially fast to zero. In the Section 4, we present some numerical results, and finally we give some concluding remarks.

2. *Existence and uniqueness of the solutions*

To stabilize the system given by (1), (2), (4)–(5), we propose the following feedback control laws: for $i = 1, 2$

$$\dot{w}_i = A_i w_i + b_i r_i(t), \quad f_i(t) = c_i^T w_i + d_i r_i(t), \tag{6}$$

where, for $i = 1, 2$, $w_i \in \mathbf{R}^{n_i}$ is the actuator state, $A_i \in \mathbf{R}^{n_i \times n_i}$ is a constant matrix, b_i, $c_i \in \mathbf{R}^{n_i}$ are constant column vectors, the superscript T stands for transpose, $d_i \in \mathbf{R}$ is a constant real number and $r_i(t)$ is defined as:

$$r_1(t) = u_t(L, t), \quad r_2(t) = \phi_t(L, t), \quad t \in \mathbf{R}. \tag{7}$$

We note that for $i = 1$ ($i = 2$, respectively) (7) and (8) give the equations for the actuator whose input is $u_t(L, t)$ ($\phi_t(L, t)$, respectively) and the output is the boundary control force $f_1(t)$ (the boundary control torque $f_2(t)$, respectively).

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† Department of Electrical and Electronics Engineering, at Bilkent University, Ankara, Turkey.

We assume the following throughout this work:

Assumptions. For $i = 1,2$.

(1) All eigenvalues of the matrix A_i are in the open left half of the complex plane,

(2) the triplet (A_i, b_i, c_i) is both observable and controllable,

(3) $d_i > 0$; furthermore for some $\gamma_i > 0$, such that $d_i > \gamma_i$, we have the following:

$$\mathcal{Re}\{h_i(j\omega)\} > \gamma_i, \quad i = 1, 2, \ \omega \in \mathbf{R}, \tag{8}$$

where $\mathcal{Re}$ denotes the real part of a complex number and for $i = 1, 2$, $h_i(s) = d_i + c_i^T(sI - A_i)^{-1}b_i$ is the actuator transfer function. □

Assumption 3 implies that, for $i = 1, 2$, the actuator transfer function $h_i(s)$ is a strictly positive real function. Let the Assumptions 1–3 stated above hold. Then, it follows from the Kalman–Yacubovitch lemma that, for $i = 1, 2$, given any symmetric positive definite matrix $Q_i \in \mathbf{R}^{n_i \times n_i}$, there exist a symmetric positive definite matrix $P_i \in \mathbf{R}^{n_i \times n_i}$ and a vector $q_i \in \mathbf{R}^{n_i}$ satisfying:

$$A_i^TP_i + P_iA_i = -q_iq_i^T - \varepsilon_iQ_i, \quad P_ib_i - \tfrac{1}{2}c_i = \sqrt{(d_i - \gamma_i)}q_i, \tag{9}$$

provided that $\varepsilon_i > 0$ is sufficiently small, see Vidyasagar (1978).

To analyze the system given by (1), (2), (4)–(7), we define the function space $\mathcal{H}$ as follows:

$$\mathcal{H} := \{(u_1\, u_2\, \phi_1\, \phi_2\, x_1\, x_2)^T) \mid u_1 \in \mathbf{H}_0^1, \\ u_2 \in \mathbf{L}^2, \ \phi_1 \in \mathbf{H}_0^1, \ \phi_2 \in \mathbf{L}^2, \ x_1 \in \mathbf{R}^{n_1}, \ x_2 \in \mathbf{R}^{n_2}\}, \tag{10}$$

where the spaces $\mathbf{L}^2$ and $\mathbf{H}_0^k$ are defined as follows:

$$\mathbf{L}^2 = \left\{f : [0, L] \to \mathbf{R} \;\middle|\; \int_0^L f^2\,\mathrm{d}x < \infty\right\}, \tag{11}$$

$$\mathbf{H}_0^k = \{f \in \mathbf{L}^2 \mid f, f', f'', \ldots, f^{(k)} \in \mathbf{L}^2, \ f(0) = 0\}. \tag{12}$$

The equations (1), (2), (4)–(8) can be written in the following abstract form:

$$\dot{z} = Az, \quad z(0) \in \mathcal{H}, \tag{13}$$

where $z = (u\, u_t\, \phi\, \phi_t\, w_1\, w_2)^T \in \mathcal{H}$, the operator $A : \mathcal{H} \to \mathcal{H}$ is a linear unbounded operator defined as

$$Ay = \begin{bmatrix} u_2 \\ \dfrac{K}{\rho}\dfrac{\partial^2 u_1}{\partial x^2} - \dfrac{K}{\rho}\dfrac{\partial \phi_1}{\partial x} \\ \phi_2 \\ \dfrac{EI}{I_\rho}\dfrac{\partial^2 \phi_1}{\partial x^2} + \dfrac{K}{I_\rho}\phi_1 - \dfrac{K}{I_\rho}\dfrac{\partial u_1}{\partial x} \\ A_1x_1 + b_1u_2(L) \\ A_2x_2 + b_2\phi_2(L) \end{bmatrix}, \tag{14}$$

where $y = (u_1\, u_2\, \phi_1\, \phi_2\, x_1\, x_2)^T$. The domain $D(A)$ of the operator A is defined as:

$$D(A) := \Big\{(u_1\, u_2\, \phi_1\, \phi_2\, x_1\, x_2)^T \mid u_1 \in \mathbf{H}_0^2, \ u_2 \in \mathbf{H}_0^1, \ \phi_1 \in \mathbf{H}_0^2, \\ \phi_2 \in \mathbf{H}_0^1, \ x_1 \in \mathbf{R}^{n_1}, \ x_2 \in \mathbf{R}^{n_2} \\ K\Big(\frac{\partial u_1}{\partial x}(L) - \phi_1(L)\Big) + c_1^Tx_1 + d_1u_2(L) = 0, \\ EI\frac{\partial \phi_1}{\partial x}(L) + c_2^Tx_2 + d_2\phi_2(L) = 0\Big\}. \tag{15}$$

It can easily be shown that $D(A)$ is dense in $\mathcal{H}$. We note that a *classical* solution of (13) is defined as an $\mathcal{H}$-valued function $z(t)$, which is continuous for $t \geq 0$, continuously differentiable and $z(t) \in D(A)$ for $t > 0$, and (13) is satisfied.

Let the Assumptions 1–3 hold, let, for $i = 1, 2$, $Q_i \in \mathbf{R}^{n_i \times n_i}$ be an arbitrary symmetric positive definite matrix and let $P_i \in \mathbf{R}^{n_i \times n_i}$, $q_i \in \mathbf{R}^{n_i}$ be the solutions of (9) where P_i is also a symmetric and positive definite matrix. In $\mathcal{H}$, we define the following inner-product:

$$\langle y, \tilde{y}\rangle_{\mathcal{H}} = \int_0^L \Big\{K\frac{\partial u_1}{\partial x}\frac{\partial \tilde{u}_1}{\partial x} \\ + \rho u_2\tilde{u}_2 + EI\frac{\partial \phi_1}{\partial x}\frac{\partial \tilde{\phi}_1}{\partial x} + I_\rho\phi_2\tilde{\phi}_2\Big\}\,\mathrm{d}x \\ + x_1^TP_1\tilde{x}_1 + \tilde{x}_1^TP_1x_1 + x_2^TP_2\tilde{x}_2 + \tilde{x}_2^TP_2x_2, \tag{16}$$

where $\tilde{y} = (\tilde{u}_1\, \tilde{u}_2\, \tilde{\phi}_1\, \tilde{\phi}_2\, \tilde{x}_1\, \tilde{x}_2)^T \in \mathcal{H}$. It can easily be shown that $\mathcal{H}$, together with the inner product defined by (16), becomes a Hilbert space.

In the sequel, we need the following inequalities:

$$u^2(s, t) \leq L\int_0^L u_x^2\,\mathrm{d}x, \quad \phi^2(s, t) \leq L\int_0^L \phi_x^2\,dx \quad s \in [0, L], \tag{17}$$

$$ab \leq \delta^2a^2 + b^2/\delta^2, \quad a, b, \delta \in \mathbf{R}, \quad \delta \neq 0, \tag{18}$$

where (17) follows from boundary conditions and Jensen's inequality, see Royden (1968).

Lemma 1. The operator A given by (14) generates a C_0-semigroup in $\mathcal{H}$, (for the terminology on semigroup theory, the reader is referred to, e.g. Pazy (1983)).

Proof. We write $A = A_U + A_B$ where,

$$A_Uy = \begin{bmatrix} u_2 \\ \dfrac{K}{\rho}\dfrac{\partial^2 u_1}{\partial x^2} \\ \phi_2 \\ \dfrac{EI}{I_\rho}\dfrac{\partial^2 \phi_1}{\partial x^2} \\ A_1x_1 + b_1u_2(L) \\ A_2x_2 + b_2\phi_2(L) \end{bmatrix}, \tag{19}$$

where $D(A_U) = D(A)$ and $y = (u_1\, u_2\, \phi_1\, \phi_2\, x_1\, x_2)^T \in \mathcal{H}$.

Using (17) and (18), it can be shown that A_B is a bounded linear operator on $\mathcal{H}$. Therefore, it is enough to show that A_U generates a C_0-semigroup on $\mathcal{H}$, see Pazy (1983). To show the latter, we use the Lumer–Phillips theorem. Hence, we have to show that for some $c > 0$, we have:

$$\langle A_Uy, y\rangle_{\mathcal{H}} \leq c\,\|y\|_{\mathcal{H}}^2, \quad y \in D(A), \tag{20}$$

where $\|\cdot\|_{\mathcal{H}}^2$ is the norm induced by (16), and for sufficiently large λ, $\lambda > c$, we have *Range of* $(\lambda I - A_U) = \mathcal{H}$ where I is the identity operator on $\mathcal{H}$.

Integrating by parts and using the boundary conditions, we obtain:

$$\langle A_Uy, y\rangle_{\mathcal{H}} = Ku_2(L)u_{1x}(L) + EI\phi_2(L)\phi_{1x}(L) \\ + x_1^T(A_1^TP_1 + P_1A_1)x_1 \\ + x_2^T(A_2^TP_2 + P_2A_2)x_2 + 2x_1^TP_1b_1u_2(L) \\ + 2x_2^TP_2b_2\phi_2(L) \leq [K\delta^2 - \gamma_1]u_2^2(L) \\ + \frac{L}{\delta^2}\int_0^L \phi_x^2\,dx \leq \frac{L}{\delta^2EI}\|z\|_{\mathcal{H}}^2, \tag{21}$$

where a first subscript denotes a component of a vector and the second subscript denotes a partial differential, i.e. $u_{1x} = \dfrac{\partial u_1}{\partial x}$, and $\delta \neq 0$ is chosen as $K\delta^2 - \gamma_1 < 0$. To obtain the first equation, we used integration by parts and (4), (5), (6) and (9), and to obtain the first inequality, we used (17) and (18).

To prove that the *Range of* $(\lambda I - A_U) = \mathcal{H}$, we show that for any $\lambda > 0$ and for any $(f_1\, f_2\, g_1\, g_2\, x_1\, x_2)^T \in \mathcal{H}$, we can find a $(u_1\, u_2\, \phi_1\, \phi_2\, z_1\, z_2)^T \in D(A)$ such that:

$$\lambda u_1 - u_2 = f_1, \quad \lambda u_2 - \frac{K}{\rho}u_{1xx} = f_2, \tag{22}$$

$$\lambda\phi_1 - \phi_2 = g_1, \quad \lambda\phi_2 - \frac{EI}{I_\rho}\phi_{1xx} = g_2, \tag{23}$$

$$(\lambda I - A_1)z_1 - b_1u_2(L) = x_1, \\ (\lambda I - A_2)z_2 - b_2\phi_2(L) = x_2. \tag{24}$$

Using the techniques presented in Kim and Renardy (1987), it can easily be shown that (22)–(24) admits a solution in $D(A)$. In fact, u_1 and ϕ_1 are given by:

$$u_1(x) = K_1 \sinh \mu_1 x - \frac{\rho}{\mu_1 K}\int_0^x [f_2(s) + \lambda f_1(s)] \sinh \mu_1(x-s)\,\mathrm{d}s, \quad (25)$$

$$\phi_1(x) = K_2 \sinh \mu_2 x - \frac{I_\rho}{\mu_2 EI}\int_0^x [g_2(s) + \lambda g_1(s)] \sinh \mu_2(x-s)\,\mathrm{d}s, \quad (26)$$

where K_1 and K_2 are constant real numbers, $\mu_1 = \lambda\sqrt{\rho/K}$, $\mu_2 = \lambda\sqrt{I_\rho/K}$. Then, u_2 and ϕ_2 are determined from (22) and (23), respectively, and the constants K_1 and K_2 are uniquely determined from $K(\phi_1(L) - u_{1x}(L)) - c_1^T z_1 - d_1 u_2(L) = 0$, $EI\phi_{1x}(L) + c_2^T z_2 + d_2\phi_2(L) = 0$. □

3. *Exponential decay of the solutions*

Let $E_B(t)$ be given by (3). We first define the following natural "energy" of the system:

$$E(t) = E_B(t) + w_1^T(t)P_1 w_1(t) + w_2^T(t)P_2 w_2(t). \quad (27)$$

It can easily be shown that for some constants $m_1 > 0$, $m_2 > 0$, the following holds:

$$m_1 \|z\|^2_{\mathscr{H}} \le E(t) \le m_2 \|z\|^2_{\mathscr{H}}. \quad (28)$$

Lemma 2. The energy $E(t)$ given by (27) is a nonincreasing function of time along the *classical* solutions of (13).

Proof. Differentiating (27) with respect to time, and using (13), we obtain:

$$\mathrm{d}E(t)/\mathrm{d}t = -\gamma_1 u_t^2(L,t) - \gamma_2\phi_t^2(L,t) - [\sqrt{d_1-\gamma_1}u_t(L,t) - w_1^T(t)q_1]^2 - [\sqrt{d_2-\gamma_2}\phi_t(L,t) - w_2^T(t)q_2]^2 - \varepsilon_1 w_1^T(t)Q_1 w_1(t) - \varepsilon_2 w_2^T(t)Q_2 w_2(t), \quad (29)$$

where to obtain (29), we differentiated (27) with respect to time, and used (1), (2), (6), then we used integration by parts, (4)–(6), and (9). Since $\mathrm{d}E(t)/\mathrm{d}t$ is nonpositive, it follows that $E(t)$ is a nonincreasing function of time along the *classical* solutions of (13). □

Next we state and prove our main result:

Theorem 1. Let $T(t)$ be the C_0-semigroup in $\mathscr{H}$ generated by A defined in (14). Then, the operator norm of $T(t)$ satisfies:

$$\|T(t)\|^2 \le M\mathrm{e}^{-\sigma t}, \quad t \ge 0, \quad (30)$$

for some positive constants M and σ.

Proof. As in Kim and Renardy (1987) and Chen (1987), we first define the following function $V(t)$:

$$V(t) = 2(1-\varepsilon)tE(t) + 2\rho\int_0^L x u_t u_x\,\mathrm{d}x + 2I_\rho\int_0^L x\phi_x\phi_t\,\mathrm{d}x + \delta I_\rho\int_0^L \phi\phi_t\,\mathrm{d}x - \delta\rho\int_0^L u u_t\,\mathrm{d}x, \quad (31)$$

where $\varepsilon \in (0,1)$ and $\delta \in (0,1)$ are constants to be determined later. To prove the assertion, we first show that for a constant $C > 0$, the following holds:

$$[2(1-\varepsilon)t - C]E(t) \le V(t) \le [2(1-\varepsilon)t + C]E(t), \quad t \ge 0. \quad (32)$$

Then we show that there exists a $T \ge 0$ such that:

$$\frac{\mathrm{d}V(t)}{\mathrm{d}t} \le 0, \quad t \ge T. \quad (33)$$

From (28), (29), (32) and (33) it follows that

$$\int_0^\infty \|z\|^4_{\mathscr{H}}\,\mathrm{d}t < \infty, \quad (34)$$

hence, (30) follows from a result due to Pazy (1983).

To prove (32), we first define the following quantities:

$$I_1 := 2\rho\int_0^L x u_t u_x\,\mathrm{d}x, \quad I_2 := 2I_\rho\int_0^L x\phi_x\phi_t\,\mathrm{d}x, \quad (35)$$

$$I_3 := \delta I_\rho\int_0^L \phi\phi_t\,\mathrm{d}x, \quad I_4 := -\delta\rho\int_0^L u u_t\,\mathrm{d}x. \quad (36)$$

Using (16), (17) and (28), we obtain the following estimates:

$$|I_1| \le 2\rho L\Big(\int_0^L u_x^2\,\mathrm{d}x + \int_0^L u_t^2\,\mathrm{d}x\Big) \le L_1 E(t), \quad (37)$$

$$|I_2| \le 2I_\rho L\Big(\int_0^L \phi_x^2\,\mathrm{d}x + \int_0^L \phi_t^2\,\mathrm{d}x\Big) \le L_2 E(t), \quad (38)$$

$$|I_3| \le \delta I_\rho\Big(\int_0^L \phi^2\,\mathrm{d}x + \int_0^L \phi_t^2\,\mathrm{d}x\Big) \le \delta I_\rho\Big(L^2\int_0^L \phi_x^2\,\mathrm{d}x + \int_0^L \phi_t^2\,\mathrm{d}x\Big) \le L_3 E(t), \quad (39)$$

$$|I_4| \le \delta\rho\Big(\int_0^L u^2\,\mathrm{d}x + \int_0^L u_t^2\,\mathrm{d}x\Big) \le \delta\rho\Big(L^2\int_0^L u_x^2\,\mathrm{d}x + \int_0^L u_t^2\,\mathrm{d}x\Big) \le L_4 E(t), \quad (40)$$

where L_1, L_2, L_3, L_4 are some positive constants. Using (37)–(40) in (31), we obtain (32) with $C = L_1 + L_2 + L_3 + L_4$.

Using integration by parts, (1)–(3), we obtain:

$$\frac{\mathrm{d}I_1}{\mathrm{d}t} = \rho L u_t^2(L,t) + KLu_x^2(L,t) - \rho\int_0^L u_t^2\,\mathrm{d}x - K\int_0^L u_x^2\,\mathrm{d}x - 2K\int_0^L x\phi_x u_x\,\mathrm{d}x, \quad (41)$$

$$\frac{\mathrm{d}I_2}{\mathrm{d}t} = I_\rho L\phi_t^2(L,t) - I_\rho\int_0^L \phi_t^2\,\mathrm{d}x + EIL\phi_x^2(L,t) - EI\int_0^L \phi_x^2\,\mathrm{d}x + 2K\int_0^L x\phi_x u_x\,\mathrm{d}x - KL\phi^2(L,t) + K\int_0^L \phi^2\,\mathrm{d}x, \quad (42)$$

$$\frac{\mathrm{d}I_3}{\mathrm{d}t} = \delta I_\rho\int_0^L \phi_t^2\,\mathrm{d}x + \delta EI\phi(L,t)\phi_x(L,t) - \delta EI\int_0^L \phi_x^2\,\mathrm{d}x + \delta K\phi(L,t)u(L,t) - \delta K\int_0^L u\phi_x\,\mathrm{d}x - \delta K\int_0^K \phi^2\,\mathrm{d}x, \quad (43)$$

$$\frac{\mathrm{d}I_4}{\mathrm{d}t} = -\delta\rho\int_0^L u_t^2\,\mathrm{d}x - \delta K u(L,t)u_x(L,t) + \delta K\int u_x^2\,\mathrm{d}x + \delta K\int u\phi_x\,\mathrm{d}x. \quad (44)$$

Using (5), (6), (17) and (18), we obtain the following estimates:

$$\int_0^L (u_x - \phi)^2\,\mathrm{d}x \le 2\int_0^L u_x^2\,\mathrm{d}x + 2L^2\int_0^L \phi_x^2\,\mathrm{d}x, \quad (45)$$

$$KLu_x^2(L,t) \le KL\phi^2(L,t) + (4L/K)(c_1^T w_1(t))^2 + (4Ld_1^2/K)u_t^2(L,t), \quad (46)$$

$$EIL\phi_x^2(L,t) \le (2L/EI)(c_2^T w_2(t))^2 + (2Ld_2^2/EI)\phi_t^2(L,t), \quad (47)$$

$$\delta EI\phi(L,t)\phi_x(L,t) \le \delta\delta_1^2 L\int_0^L \phi_x^2\,\mathrm{d}x + (2\delta/\delta_1^2)(c_2^T w_2(t))^2 + (2\delta d_2^2/\delta_1^2)\phi_t^2(L,t), \quad (48)$$

$$\delta K[\phi(L,t) - u_x(L,t)]u(L,t) \le \delta\delta_2^2 L\int_0^L u_x^2\,\mathrm{d}x + (2\delta/\delta_2^2)(c_1^T w_1(t))^2 + (2\delta d_1^2/\delta_2^2)u_t^2(L,t), \quad (49)$$

where $\delta_1 \neq 0$, $\delta_2 \neq 0$ are arbitrary constants. Differentiating (31) with respect to time, using (29), (41)–(49), and collecting likewise terms, we obtain:

$$\begin{aligned}\frac{dV(t)}{dt} \leq & -[2(1-\varepsilon)t\varepsilon_1 w_1^T(t)Q_1 w_1(t) - 2(1 \\ & -\varepsilon)w_1^T(t)P_1 w_1(t) - ((4L/K) \\ & + (2\delta/\delta_2^2))(c_1^T w_1(t))^2] \\ & - [2(1-\varepsilon)t\varepsilon_2 w_2^T(t)Q_2 w_2(t) \\ & - 2(1-\varepsilon)w_2^T(t)P_2 w_2(t) - ((2L/EI) \\ & + (2\delta/\delta_1^2))(c_2^T w_2(t))^2] \\ & - (\varepsilon+\delta)\rho \int_0^L u_t^2 \, dx - [(\varepsilon+\delta)EI \\ & + (2\varepsilon+\delta-3)KL^2 - \delta\delta_1^2 L] \int_0^L \phi_x^2 \, dx \\ & - (\varepsilon-\delta)I_\rho \int_0^L \phi_t^2 \, dx - [(2\varepsilon-\delta-1)K \\ & - \delta\delta_2^2 L] \int_0^L u_x^2 \, dx \\ & - 2(1-\varepsilon)t[\sqrt{d_1-\gamma_1}u_t(L,t) - w_1^T(t)q_1]^2 \\ & - 2(1-\varepsilon)t[\sqrt{d_2-\gamma_2}\phi_t(L,t) - w_2^T(t)q_2]^2 \\ & - [2(1-\varepsilon)t\gamma_1 - \rho L - (4Ld_1^2/K) \\ & - (2\delta d_1^2/\delta_2^2)]u_t^2(L,t) \\ & - [2(1-\varepsilon)t\gamma_2 - I_\rho L - (2Ld_2^2/EI) \\ & - (2\delta d_2^2/\delta_1^2)]\phi_t^2(L,t). \end{aligned} \tag{50}$$

Let us choose ε, δ, $\in [0, 1]$, and $\delta_1 \neq 0$, $\delta_2 \neq 0$ sufficiently small such that:

$$\frac{\delta}{1-\delta} > \frac{KL^2}{EI}, \quad \varepsilon - \delta > 0,$$
$$(2\varepsilon - \delta - 1)K - \delta\delta_2^2 L > 0,$$
$$2\varepsilon - \delta - 1 > 0,$$
$$(\varepsilon+\delta)EI + (2\varepsilon+\delta-3)KL^2 - \delta\delta_1^2 L > 0.$$

The above inequalities always have a solution, for example by choosing $\delta = (EI + 2KL^2)/(2EI + 2KL^2)$, $\varepsilon = (7EI + 8KL^2)/(8EI + 8KL^2)$, and then $\delta_1^2 < EI(11EI + 10KL^2)/(8\delta LEI + 8\delta KL^3)$ and $\delta_2^2 < EIK/(4\delta LEI + 4\delta KL^3)$, we see that the above inequalities are satisfied. Then, from (50) it follows that there exists a $T \geq 0$ depending only on ε, δ, δ_1, δ_2, ε_1, ε_2 and the coefficients in (1), (2), such that (33) holds. Therefore, using the argument following (33), we arrive at (30). □

Remark 1. In case of a non-dynamic feedback, (6) reduces to:

$$f_1(t) = d_1 u_t(L, t), \quad f_2(t) = d_2 \phi_t(L, t), \tag{51}$$

which is the case considered in Kim and Renardy (1987). □

Remark 2. We note that both of the actuator transfer functions given by the control laws (6) and (51) are not band-limited, see (8). Also, the restriction of actuator transfer functions to positive real functions might seem to be too restrictive, but nevertheless this type of actuator transfer functions are more general than those given by (51). Moreover, the proposed control laws are more suitable for some control applications, such as eigenvalue assignment, disturbance rejection, etc. than the control law given by (51). This point will be explained below and in Section 4.

One way of implementing the control laws given by (51) is to use actuators whose inputs are $u_t(L, t)$ and $\phi_t(L, t)$, and whose outputs are $f_1(t)$ and $f_2(t)$, where the actuator transfer functions are given by d_1 and d_2, respectively. From a practical point of view, however, most actuators show some dynamic behaviour, at least over a frequency range, hence their transfer functions are not *constants*. In this case, Theorem 1 provides a sufficient condition to ensure exponential stability, whereas the results of Kim and Renardy (1987) do not apply.

Also note that the proposed dynamic control, (6), as well as the standard non-dynamic one, (51), change the frequency-domain characteristic of the uncontrolled system; that is the eigenvalues of the operator given by (14) are completely different from the eigenvalues of the uncontrolled system. This change in the spectrum, although limited, may possibly be used for some control applications, such as eigenvalue assignment, disturbance rejection, etc. Note that the dynamic control offers more degrees of freedom to change the spectrum of the operator given by (14), than the standard non-dynamic one. For example, simulation studies of Chen (1987) and Kim and Renardy, (1987), show that the non-dynamic boundary control affects the eigenvalue of the system *uniformly*, whereas by using dynamic boundary control, it may be possible to change the spectrum of the system only over a prescribed frequency range, without changing the rest of the spectrum very much. In the next section, we give some simulation results indicating this use of dynamic boundary control. □

4. *Numerical results*

In this section, to show the effect of the proposed control laws given by (6), on the eigenvalues of the system given by (14), we present some numerical simulation results. We use normal mode analysis and set:

$$\begin{aligned} u(x,t) &= e^{\lambda t}U(x), \\ \phi(x,t) &= e^{\lambda t}\Phi(x), \\ w_i(t) &= e^{\lambda t}W_i, \; i = 1, 2, \end{aligned} \tag{52}$$

where $x \in [0, L]$, $t \geq 0$, U, $\Phi: [0, L] :\to \mathbf{R}$ are appropriate functions, $W_i \in \mathbf{R}^{n_i}$, $i = 1, 2$ are appropriate constant vectors and λ is the eigenvalue to be determined. To find the eigenvalues satisfying (52) and (1), (2), (4)–(6), we use finite difference technique with N point spatial discretization, approximating the spatial derivatives by using a central difference formula, see Greenspan and Casulli (1988). The resulting equations can be rewritten in the form $\det(\lambda^2 P + \lambda Q + R) = 0$ where P, Q, $R \in \mathbf{R}^{m \times m}$, $m = 2N + n_1 + n_2$, are appropriate matrices. This equation takes on the customary form: $A = \lambda B$ where A, $B \in \mathbf{R}^{2m \times 2m}$ are given as:

$$A = \begin{bmatrix} 0 & -R \\ I & Q \end{bmatrix}, \quad B = \begin{bmatrix} -I & 0 \\ 0 & -P \end{bmatrix}. \tag{53}$$

In the simulations we use the following set of parameters which model a solid aluminium bar: $\rho = 40 \text{ kg m}^{-1}$, $K = 2.8 \times 10^8 \text{ kgm sec}^{-2}$, $EI = 6.3 \times 10^5 \text{ sec}^{-2}$, $I_\rho = 0.0332 \text{ kgm}$, $L = 2 \text{ m}$.

To see the difference between the effects of the non-dynamic and the dynamic boundary control on the eigenvalues we first consider the non-dynamic boundary control as given by (51) with the following parameters:
Case 1: $d_1 = 1$, $d_2 = 1$.

By comparing the results of $N = 50$ and $N = 55$, we conclude that about 27 complex conjugate pairs have converged at $N = 50$, with relative error of both real and imaginary parts of the eigenvalues less than 1%. Due to space limitation, we present only the first five lowest eigenvalues. Note that all converged eigenvalues have negative real parts which decrease (i.e. damping increases) as the imaginary parts increase, in accordance with the results of Kim and Renardy (1987).

For some control applications it may be desirable to change the spectrum only over a prescribed frequency range. For example, the beam may be subject to a disturbance with a known frequency context. In this case, to reduce the effect of the disturbance, it may be desirable to introduce more damping only to the modes of the beam over the frequency range of the disturbance.

From Table 1, we see that the higher modes are damped rather well, but the damping associated with the lower modes, especially the first and second modes, are rather small. Therefore we may want to introduce more damping to

TABLE 1. EIGENVALUES FOR CASE 1

$1.0 \times 10^5 \bigstar$
$-0.00000043228545 \pm 0.00119485753223i$
$-0.00000192237411 \pm 0.00740277945165i$
$-0.00000449568932 \pm 0.02036214818773i$
$-0.00000806812670 \pm 0.03892750846202i$
$-0.00001229204976 \pm 0.06244736533499i$

TABLE 2. EIGENVALUES FOR CASE 2

$1.0 \times 10^5 \bigstar$
$-0.00000516460452 \pm 0.00119485857851i$
$-0.00000192796170 \pm 0.00740309743653i$
$-0.00000449362672 \pm 0.02036241081321i$
$-0.00000807053831 \pm 0.03892776239342i$
$-0.00001228631474 \pm 0.06244758839125i$

TABLE 3. EIGENVALUES FOR CASE 3

$1.0 \times 10^5 \bigstar$
$-0.00000086525939 \pm 0.00119485760277i$
$-0.00000197409347 \pm 0.00740308902514i$
$-0.00000450748096 \pm 0.02036241038715i$
$-0.00000807875856 \pm 0.3892775964977i$
$-0.00001228868545 \pm 0.06244759418429i$

TABLE 4. EIGENVALUES FOR CASE 4

$1.0 \times 10^5 \bigstar$
$-0.00000087102608 \pm 0.00119485782482i$
$-0.00000192340624 \pm 0.00740281189987i$
$-0.000004492621268 \pm 0.02036217141781i$
$-0.00000807099807 \pm 0.03892753940502i$
$-0.00001228458524 \pm 0.06244738387793i$

TABLE 5. EIGENVALUES FOR CASE 5

$1.0 \times 10^5 \bigstar$
$-0.00005969479095 \pm 0.00119290900150i$
$-0.00000197299698 \pm 0.00740596154149i$
$-0.00000450397047 \pm 0.02036479425832i$
$-0.00000807341702 \pm 0.03892998790025i$
$-0.0000122828955 \pm 0.06244970471820i$

TABLE 6. EIGENVALUES FOR CASE 6

$1.0 \times 10^5 \bigstar$
$-0.00000043346442 \pm 0.00119478592758i$
$-0.00002239037315 \pm 0.00740285578075i$
$-0.00000456806342 \pm 0.02036401853819i$
$-0.00000809900202 \pm 0.03892910667506i$
$-0.00001229928068 \pm 0.06244883025543i$

TABLE 7. EIGENVALUES FOR CASE 7

$1.0 \times 10^5 \bigstar$
$-0.00000355607727 \pm 0.00119483237520i$
$-0.00001914733864 \pm 0.00740290908041i$
$-0.00000456294907 \pm 0.02036392203410i$
$-0.00000809851705 \pm 0.03892905890156i$
$-0.00001229725497 \pm 0.06244880700766i$

the first two modes without changing much the remaining ones. Note that this could be achieved by increasing d_1 and d_2, but in that case the remaining modes are also affected uniformly (see Kim and Renardy (1987)), and the required actuator energy will possibly increase, which may cause saturation in the actuator.

To introduce more damping only to the lower modes, we propose the following actuator transfer function:

$$h_i(s) = d_i + \frac{K_i s}{s^2 + 2\xi_i\omega_{0i}s + \omega_{0i}^2}, \quad i = 1, 2, \tag{54}$$

where, for $i = 1, 2$, K_i, ξ_i and ω_{0i}^2 are positive constants.

The real part of $h_i(j\omega)$ is given by:

$$\mathcal{Re}\{h_i(j\omega)\} = d_i + \frac{2K_i\xi_i\omega_{0i}\omega^2}{(\omega_{0i}^2 - \omega^2)^2 + 4\xi_i^2\omega_{0i}^2\omega^2}, \quad i = 1, 2, \tag{55}$$

hence (8) is satisfied with $\gamma_i = d_i$, for $i = 1, 2$. The maximum of $\mathcal{Re}\{h_i(j\omega)\}$ is obtained at $\omega = \omega_{0i}$ and is given by:

$$\max_{\omega\in\mathbf{R}} \mathcal{Re}\{h_i(j\omega)\} = d_i + \frac{K_i}{2\xi_i\omega_{0i}}, \quad i = 1, 2, \tag{56}$$

and $\mathcal{Re}\{h_i(j\omega)\}$ decreases to d_i as $\omega \to 0$ and $\omega \to \infty$. Also note that the proposed dynamic controller does not increase uniformly the minimum of $\mathcal{Re}\{h_i(j\omega)\}$, that is;

$$\inf_{\omega\in\mathbf{R}} \mathcal{Re}\{h_i(j\omega)\} = d_i, \quad i = 1, 2. \tag{57}$$

Since we want to decrease $\mathcal{Re}\{\lambda_1\}$ and $\mathcal{Re}\{\lambda_2\}$, where λ_1 and λ_2 are first and second eigenvalues in Table 1, respectively, from the reasoning above we conclude that a good choice for this purpose is $\omega_{0i} = \mathcal{Im}\{\lambda_i\}$ or $\omega_{0i} = \mathcal{Im}\{\lambda_2\}$. For $N = 50$, we calculated the eigenvalues for the following choices of the actuator parameters for $i = 1, 2$

Case 2: $d_i = 1$, $K_1 = 119.48$, $\xi_i = 0.05$, $\omega_{0i} = 119.48$,
Case 3: $d_i = 1$, $K_i = 119.48$, $\xi_i = 0.5$, $\omega_{0i} = 119.48$,
Case 4: $d_i = 1$, $K_i = 11.948$, $\xi_i = 0.05$, $\omega_{0i} = 119.48$,
Case 5: $d_i = 1$, $K_i = 1194.8$, $\xi_i = 0.05$, $\omega_{0i} = 119.48$,
Case 6: $d_i = 1$, $K_i = 740.28$, $\xi_i = 0.05$, $\omega_{0i} = 740.48$,
Case 7: $d_i = 1$, $K_1 = 119.48$, $K_2 = 740.28$, $\xi_i = 0.05$, $\omega_{01} = 119.48$, $\omega_{02} = 740.28$.

In all cases, the five lowest eigenvalues associated with the beam vibrations are given in the Tables 2–7, respectively.

As it can be seen from Tables 2–6, the resonant frequency ω_{0i} of the actuator determines the frequency of the mode which is affected by the control law; decreasing the damping constant ξ_i and/or increasing the gain K_i have increasing effect on the damping of the mode of interest, as expected from (56). Also from Table 7, we see that it may be possible to introduce more damping only to the lowest modes by using different actuators for force and moment control.

The results of these simulations suggest that it might be possible to change the spectrum of the system given by (1)–(5) over a specified frequency range, without altering the remaining part of the spectrum very much. This point, as well as possible applications of the ideas presented in this section, need further investigation.

5. *Conclusion*

In this paper we considered the stabilization of a clamped-free Timoshenko beam using dynamic boundary control. Under some assumptions, one of which is the positive realness of the actuator transfer functions corresponding to the dynamic boundary controls, we proved that the energy of the actuator-beam configuration decays exponentially to zero. We also give some numerical simulation results. These simulations results suggest that it might be possible to change the spectrum of the system over a specified frequency range by using dynamic boundary control without affecting the remaining part of the spectrum. Previous numerical simulation results suggest that one may not obtain such a result by using only non-dynamic feedback, see Chen (1987) and Kim and Renardy (1987). This change in the spectrum could be used in some control applications, such as eigenvalue assignment, disturbance rejection, etc.

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